

Some remarks on Tits geometries

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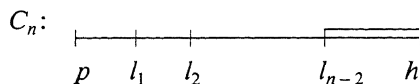
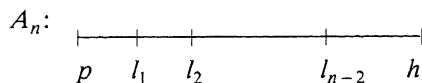
ABSTRACT

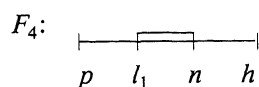
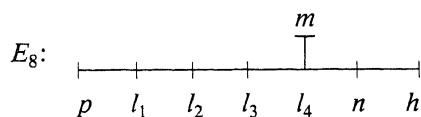
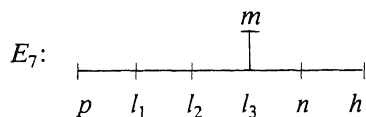
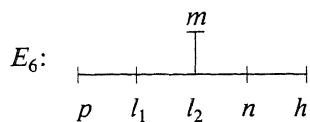
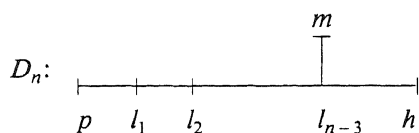
A result of Tits' in his paper "A local approach to buildings" is somewhat strengthened: it is shown that each geometry of type D_n or E_6 is a building. A counterexample to the corresponding statement for E_7 is given. Moreover, a proof is given of the fact that any thick finite geometry of spherical type all of whose residues of type C_3 are buildings is itself a building.

In [10] Tits proves the following

1. THEOREM. *Let G be a geometry of type $M = A_n, B_n (= C_n), D_n, E_6, E_7, E_8,$ or F_4 . Then G is a building if and only if it has the following properties: if $M = C_n, D_n$ or E_6 , properties (O) and (LL), if $M = E_7$, properties (O), (LL) and (LH), if $M = E_8$ or F_4 , properties (O), (LL), (LH) and (HH).*

For notation and terminology, the reader is referred to [10]. We only recall the meaning of the properties (O), (LL), (LH), (HH). For this we need the following labelling of the diagrams:





The elements of type p , l_i and h are called *points*, *lines* and *hyperlines* respectively.

- (O) If two elements of type l_i (for some i) have the same shadow in the set of all points, they coincide.
- (LL) If two lines are both incident to two distinct points, they coincide.
- (LH) If a line and a hyperline are both incident to two distinct points, they are incident.
- (HH) If two distinct hyperlines are both incident to two distinct points, the latter are incident to a line.

Here we show that any geometry of type D_n or E_6 is a building. For thick finite geometries of type D_n , this is stated in Timmesfeld [7], where the case $n = 4$ is attributed to Th. Meixner; the proof in [loc. cit.], however, is valid in general. Also, Tits [11] has observed that thick finite geometries of type A_n , D_n , E_6 or E_7 are buildings on the basis of a case-by-case argument. Our final proposition extends this observation to geometries of arbitrary spherical type whose residues of type C_3 are buildings by use of a unified proof, valid in all cases.

2. COROLLARY. (a) *Every geometry of type A_n , D_n or E_6 is a building.*
 (b) *A geometry of type E_7 is a building if and only if it satisfies (LH).*
 (c) *A geometry of type E_8 is a building if and only if it satisfies (LH) and (HH).*

PROOF. By the above theorem, every geometry of type A_n is a building. In order to apply Tits' theorem to the other cases, we prove (O) and (LL) for D_n , E_6 and E_7 , and – assuming (LH) – also for E_8 .

For the sake of completeness we shall repeat the argument given by Timmesfeld [7] in case of D_n .

First, we introduce some terminology.

If v_1, v_2, \dots, v_t are elements of a geometry, we say that $v_1 - v_2 - \dots - v_t$ is a chain whenever v_i and v_{i+1} are incident for each i ($1 \leq i < t$).

For any object X the set of points in $\text{Res}(X)$ is called the *point shadow* of X , and the set of lines in $\text{Res}(X)$ is called the *line shadow* of X .

We may assume $n > 3$.

Step 1. *If every geometry of type D_{n-1} is a building, then any geometry of type D_n satisfies (LL).*

Given lines L and L_1 both incident with two distinct points P and P_1 , we can find a chain $L - M - H - L_1$, where M, H have types m, h respectively. (Such a chain exists, as may be seen, e.g., in $\text{Res}(P)$ which is a building of type D_{n-1} by assumption.)

Now $\{M, H\}$ is a flag, and $\text{Res}(\{M, H\})$ is a projective space containing P and P_1 and hence also a line L_2 incident with P and P_1 .

Moreover, $\text{Res}(M)$ is a projective space containing P, P_1 and the lines L, L_2 incident to both points, whence $L = L_2$. Similarly for $\text{Res}(H)$, we see that $L_1 = L_2$, so that $L = L_1$.

Step 2. *If a geometry of type D_n, E_6, E_7 or E_8 satisfies (LL) and if $\text{Res}(P)$ satisfies (O) for each point P , then the geometry itself satisfies (O).*

Suppose M and M_1 are both objects of type l_i for some i with the same point shadow S . Let L be a line incident to M .

Take distinct points P and P_1 incident to L . Then P and P_1 belong to S , so there is a line L_1 incident to P, P_1 and M_1 .

In view of (LL), we have $L = L_1$, so L belongs to the line shadow of M_1 . Thus the line shadow of M_1 contains the line shadow of M , and by symmetry, the line shadows of M and M_1 coincide. Consequently, the point shadows of M and M_1 in $\text{Res}(P)$ coincide. Since (O) holds for $\text{Res}(P)$ by assumption, M and M_1 coincide.

Step 3. *Statement (a) holds.*

For the case D_n , this is immediate from the two previous steps by induction on n .

Thus, consider a geometry G of type E_6 . In view of the theorem, Step 2 and statement (a), we need to show (LL).

Let L and L_1 be lines, both incident to the distinct points P and P_1 . Since

$\text{Res}(P)$ is a building of type D_5 , we can find a hyperline H of G in $\text{Res}(P)$ incident to both L and L_1 . But $\text{Res}(H)$ is a building of type D_5 , too, so satisfies (LL) . Since P and P_1 (being incident to L) are elements of $\text{Res}(H)$, this yields $L = L_1$.

Step 4. *Every geometry of type E_7 satisfies (O) and (LL). In particular, statement (b) holds.*

In view of the two previous steps, we need only verify property (LL) for a geometry of type E_7 .

Given lines L and L_1 both incident to two distinct points P and P_1 , we can find a hyperline H in $\text{Res}(P)$ which is incident to both L and L_1 . (For $\text{Res}(P)$ is a building of type E_6 .)

Since P and P_1 are both elements of the building $\text{Res}(H)$ of type D_6 , it follows that $L = L_1$.

Step 5. *Let G be a geometry of type E_8 . If G satisfies (LL), then (O) holds. If G satisfies (LH), then (LL) holds. In particular, statement (c) holds.*

The first part follows from Step 2, Step 4 and the theorem. Suppose G has property (LH) . Let P and P_1 be two distinct points, both incident to the lines L and L_1 . Take a hyperline H incident to L_1 . Then P and P_1 are also incident to H , so applying (LH) to L and H , we find that L is incident to H . But $\text{Res}(H)$ is a building due to (a), hence satisfies (LL) . It follows that $L = L_1$.

This ends the proof of Step 5, and hence the proof of the corollary.

3. REMARKS. (i). Using a result of Buekenhout and Shult [1, 2] and an older (elementary) result of Tits [8] 7.3n, we can prove that a geometry of type D_n is a building without recourse to the above theorem.

(ii). For a geometry of type E_8 the properties (LH) and (HH) can be replaced by the following property:

For each pair of distinct hyperlines, both incident to two distinct points, there is a line incident to both points and both hyperlines.

4. EXAMPLE. We exhibit a quotient of the unique building G of type E_7 over the field \mathbb{C} of complex numbers by a group of order 2, which is a geometry of type E_7 but not a building. This shows that Condition (LH) in the corollary is not superfluous. The example is an analogue of the ones given by Tits [10] for buildings of type C_n , and is given in terms of Ferrar's presentation [5] of E_7 .

Let J_1 be the exceptional 27-dimensional Jordan algebra over the field \mathbb{R} of the real numbers with positive definite trace form, and denote by J its complexification, by t complex conjugation with respect to the real form J_1 , by $\langle \cdot, \cdot \rangle$ the standard bilinear form on J (positive definite on J_1), by $N(\cdot)$ the standard cubic form, by $\langle \cdot, \cdot, \cdot \rangle$ its linearization, and by $*$ the cross product such that $\langle A * B, C \rangle = 6 \langle A, B, C \rangle$ for A, B, C in J .

The ternary algebra M_1 is the 56-dimensional real vector space $\mathbb{R} + \mathbb{R} + J_1 + J_1$, supplied with the alternating bilinear form $\{.,.\}$ given by

$$\{x_1, x_2\} = a_1 b_2 - a_2 b_1 + \langle A_1, B_2 \rangle - \langle A_2, B_1 \rangle$$

and the symmetric trilinear product $(x_1, x_2, x_3) \mapsto x_1 x_2 x_3$ obtained by linearizing the expression

$$\begin{aligned} xxx &= 6(-a^2 b + a \langle A, B \rangle - 2N(b)) & , \\ & b^2 a - b \langle A, B \rangle + 2N(a) & , \\ & (ab - \langle A, B \rangle)A - bB^*B + B^*(A^*A), \\ & (-ab + \langle A, B \rangle)B + aA^*A - A^*(B^*B) & , \end{aligned}$$

where $x = (a, b, A, B)$ and $x_i = (a_i, b_i, A_i, B_i)$ in M_1 for each i (thus a_i, b_i in \mathbb{R} and A_i, B_i in J_1).

Then $\text{Aut } M_1$ is a real Lie group of type E_7 . Denote by M the complexification of M_1 and retain the notation for the linear extensions to M of the bilinear form and the ternary multiplication on M_1 .

Consider the subset S of M consisting of all members of M for which xxM is contained in $\mathbb{C}x$.

Instead of $\mathbb{C}x$, we shall also write $\langle x \rangle$. For x_1, x_2 in M with $\langle x_1 \rangle, \langle x_2 \rangle$ distinct elements of S , we call $\langle x_1 \rangle, \langle x_2 \rangle$ *adjacent* – notation $\langle x_1 \rangle \sim \langle x_2 \rangle$ – whenever $Mx_1 x_2$ is contained in $\mathbb{C}x_1 + \mathbb{C}x_2$. For details on M and S , the reader is referred to Faulkner [4] and Ferrar [5].

We are interested in the graph (S, \sim) . It has diameter 3, and two vertices $\langle x_1 \rangle$ and $\langle x_2 \rangle$ have distance at most 2 if and only if $\{x_1, x_2\} = 0$. Moreover, (S, \sim) is isomorphic to the graph (S', \sim') obtained from the building G by letting S' be the set of points of G , and letting $P \sim' Q$ for distinct P, Q in S' stand for the existence of a line of G incident to both P and Q , i.e., for *collinearity* of P and Q .

It is known [3] that G can be uniquely reconstructed from (S, \sim) up to isomorphism. Thus, any automorphism of (S', \sim') extends uniquely to an automorphism of G .

From now on we shall identify (S, \sim) and (S', \sim') .

Consider the semilinear transformation f of M given by

$$x^f = (-\bar{b}, \bar{a}, -B^t, A^t), \text{ where } x = (a, b, A, B) \text{ in } M.$$

The transformation f preserves S and induces an automorphism $f|S$ of (S, \sim) of order two such that P has distance 3 to P^f in (S, \sim) for any vertex P of S . It readily follows that the unique automorphism of G extending $f|S$ satisfies condition (Q3) of [10] and that the group F of order two which it generates acts freely on the set of all flags of G of corank 2.

Hence, by [loc. cit.], the quotient G/F is a geometry of type E_7 , but not a building (e.g., since there are points in this quotient which are collinear to exactly two points of a line).

In order to deal with thick finite geometries we need the following lemma.

5. LEMMA. Let s be a natural number. Suppose S is a finite regular connected graph with v points, valency $k \equiv 0 \pmod{s}$ and diameter d such that for all i ($1 \leq i \leq d$) and all x, y in S with mutual distance $d(x, y) = i$ we have

$$\# \{z \in S \mid 1 = d(x, z) = i - d(y, z)\} \equiv 1 \pmod{s},$$

and

$$\# \{z \in S \mid 1 = d(x, z) = d(y, z) - i\} \equiv 0 \pmod{s}.$$

Then

(i) Each eigenvalue w of the adjacency matrix A of S distinct from k satisfies $w \equiv -1 \pmod{p}$ for every maximal ideal p containing s of the ring of algebraic integers generated by the eigenvalues of A . Moreover, $v \equiv 1 \pmod{s}$.

(ii) If g is an automorphism of S without fixed points such that each point in S is not adjacent to its image under g , then $s = 1$.

PROOF. (i). By induction with respect to e , one can show that for every $e \geq 0$ the x, y entry of $(A + I)^e$ equals 0 if $d(x, y) > e$ and equals 1 \pmod{s} if $d(x, y) \leq e$. Thus, there is a matrix B with integral coefficients such that

$$(*) \quad (A + I)^d = J + sB,$$

where J is the 'all 1' matrix.

Since S is regular of valency k , the matrices A , J , I and B commute and the 'all 1' vector j is a common eigenvector.

By (*), we get $(k + 1)^d j = (v + sj)j$ for some integer i , and hence that $(k + 1)^d \equiv v \pmod{s}$. It follows that $v \equiv 1 \pmod{s}$ as $k \equiv 0 \pmod{s}$ by assumption. This proves the last statement of (i).

Since A and J are symmetric and S is connected, all other common eigenvectors are real and orthogonal to j (with respect to the standard inner product). Let w be an eigenvalue of A distinct from k corresponding to the common eigenvector b , say. Then $Jb = 0$, so by (*), we get

$$(w + 1)^d b = (sm)b \text{ for some algebraic integer } m.$$

Consequently, $w + 1 \equiv 0 \pmod{p}$ for each maximal ideal p containing s , whence (i).

(ii). Let M denote the permutation matrix of g on the points of S (considered as the basis of a real vector space). Suppose that p is a maximal ideal containing s . Now, A and M commute as g is an automorphism of S , so in view of (i) the matrix $(A + I)M$ has eigenvalues which are all 0 \pmod{p} except for the eigenvalue $(k + 1)$ of multiplicity 1 with eigenvector j . As $k \equiv 0 \pmod{p}$, we get $\text{trace}((A + I)M) \equiv 1 \pmod{p}$. On the other hand, this trace equals 0 according to the assumption that no point is adjacent to or coincides with its image under g . This contradiction shows that s does not belong to any maximal ideal, whence $s = 1$ and we are done.

6. REMARK. Let M be one of the diagrams in Theorem 1. Consider a thick finite building of type M . It can easily be checked that the (collinearity) graph

S on the set of points in which two vertices x, y are adjacent if and only if there is a line incident to both x and y , satisfies the hypotheses of the above lemma, with s a power of the characteristic of the field over which the building is defined. In particular, $s > 1$.

7. REMARK. We are indebted to F. Timmesfeld for pointing out to us the existence of a thick finite geometry of type C_3 associated with the alternating group on 7 letters, which is not covered by a building (in fact, the corresponding chamber system is 2-connected), see Kantor [6].

The involution exchanging opposite vertices of the octahedron (the thin building of type C_3) yields a quotient geometry of type C_3 which is not a building. In view of the existence of these geometries, the following result is to a certain extent best possible.

8. PROPOSITION. *Any thick finite geometry of spherical type all of whose residues of type C_3 are covered by buildings is a building.*

PROOF. For the types A_n , D_n and E_6 this is immediate from the corollary. However, we shall not make use of this.

Let G be a thick finite geometry of spherical type M . Without loss of generality, we may assume that M is connected.

If $n \leq 2$, there is nothing to prove.

Suppose that $n \geq 3$.

If $M = H_3$ or H_4 , the statement is easily verified by use of the Feit-Higman Theorem.

Since each residue of type C_3 is covered by a building, Theorem 1 of [10] yields that $G = D/A$ (up to isomorphism), where D is a building of type M and A is a group of automorphisms of D acting freely on the set of all flags of corank 2 and satisfying (Q1) of [10]. Denote by $p: D \rightarrow G$ the canonical projection.

Let X be an element of D . By induction on n , the residue of X^p is a building, so by [10] 6.1.8, the restriction of p to $\text{Res}(X)$ is an isomorphism onto $\text{Res}(X^p)$. Consequently, $\text{Res}(X)$ is a finite building, and so is D by [9]. By (Q1), the restriction of p to $\text{Res}(X)$ is induced by the quotient map with respect to the stabilizer of X in A . Thus, any automorphism in A fixing X fixes every element of $\text{Res}(X)$, and, by connectedness of D , is the identity.

Suppose A contains a nontrivial automorphism a .

According to (Q2') of [10] – a consequence of (Q1) – there is no element Y of D incident to both X and X^a . In particular, for each point P of D its image P^a is not collinear with P . This, however, contradicts (ii) of Lemma 5 in view of Remark 6. It follows that A is trivial, and that $G = D$ is a building.

9. REMARK. In the above proposition for geometries of type C_n we can weaken the 'thick' part of the hypothesis by only requiring that lines are incident to at least three points.

10. ACKNOWLEDGEMENT

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Appendix

On the distance between opposite vertices in buildings of spherical type

by J. Tits

In [LA], 1.4, example (b), I indicated how, starting from a compact Lie group of type B_n (and also from certain non compact groups), one can construct geometries of type B_n which are not buildings (they are not simply connected). In the above paper (§ 4), a similar construction is described in the case of E_7 . The main purpose of this appendix is to show, by a uniform proof, that the same method applies to a compact simple Lie group G if and only if the connection index (order of the fundamental group = order of the centre of the universal covering) of G is ≤ 2 ; in other words, excluded are only the types A_n ($n \geq 2$), D_n , E_6 . This shows that Corollary 2 (a) of the above paper is, in a certain sense, “best possible”.

Let Φ be a reduced irreducible root system in a real vector space V (we suppose V spanned by Φ), let $A = \{a_1, \dots, a_l\}$ be a basis of Φ and let $d = \sum_{i=1}^l d_i a_i$ denote the highest root. The following lemma is well known.

LEMMA. Let $i \in \{1, \dots, l\}$, and let a_{i_1}, \dots, a_{i_m} denote the elements of A which are connected with a_i in the Dynkin diagram. Then, $d_i \geq \frac{1}{2} \sum_{j=1}^m d_{i_j}$.

Let $f: V \rightarrow \mathbb{R}$ be the coroot associated with a_i , that is, $f(v) = 2(a_i, v)/(a_i, a_i)$. The number $f(a_k)$ is equal to 2 if $k=i$, is a strictly negative integer if $k \in \{i_1, \dots, i_m\}$, and is zero otherwise. Since $d+a_i$ is not a root, we have

$$0 \leq f(d) = 2d_i + \sum_{j=1}^m f(a_{i_j})d_{i_j} \leq 2d_i - \sum_{j=1}^m d_{i_j},$$

hence the claim.

PROPOSITION. For $i, j \in \{1, \dots, l\}$, with $i \neq j$, set $\Phi(i, j) = \{ \sum c_k a_k \in \Phi \mid c_i \neq 0, c_j \neq 0 \}$ and let $V(i, j)$ denote the subspace of V spanned by $\Phi(i, j)$. Then, one has $V(i, j) \neq V$ if and only if $d_i = d_j = 1$.

If $d_i = d_j = 1$, we have

$$V(i, j) \subset \mathbb{R} \cdot (a_i + a_j) + \sum_{k \neq i, j} \mathbb{R} \cdot a_k \neq V.$$

To prove the converse, we assume, without loss of generality, that $i=1$, that $d_j > 1$ and that (a_1, a_2, \dots, a_j) is the unique chain joining a_1 and a_j on the Dynkin diagram (i.e., that a_k and a_{k+1} are connected in that diagram for $1 \leq k \leq j-1$). Set $b = a_1 + \dots + a_j$ and let $b_0 = b, b_1, \dots, b_m = d$ be a sequence of roots such that $b_r - b_{r-1} \in A$ for $1 \leq r \leq m$. Clearly, $V(i, j)$ contains all b_r ($0 \leq r \leq m$), hence also the set $A' = \{b_r - b_{r-1} \mid 1 \leq r \leq m\}$. The smallest integer $j' \in [1, j]$ such that $d_k \geq 2$ whenever $j' \leq k \leq j$ must be 1 or 2, otherwise the above lemma would imply $1 = d_{j'-1} > \frac{1}{2}d_{j'}$. Consequently, a_2, a_3, \dots, a_j belong to A' and so do of course a_{j+1}, \dots, a_l . Therefore,

$$V(i, j) \supset \mathbb{R} \cdot b + \sum_{k \neq 1} \mathbb{R} \cdot a_k = V,$$

q.e.d.

Let Σ denote the Coxeter complex associated with Φ , that is, ‘‘cut out’’ on the unit sphere of V (identified with its dual) by the kernels of the roots. Let Δ be a building of type Σ : here, ‘‘weak buildings’’ in the sense of [BN] are simply called ‘‘buildings’’, so that we allow Δ to be $= \Sigma$. Remember that two vertices of Δ are said to be in ‘‘generic position’’ if they belong to opposite chambers.

COROLLARY 1. Two vertices p, q of Δ of nonopposite types i and j and in generic position are at distance at least two in the graph of vertices of Δ . They are at distance exactly two if and only if $d_i = d_j = 1$.

By [BN], 3.9, it suffices to consider the case where $\Delta = \Sigma$. Let q' be the vertex of Σ opposite to q and let j' be its type. We assume, without loss of generality, that p and q' are the vertices of types i and j' of the ‘‘fundamental chamber’’

corresponding to the basis A . Clearly, the kernels of all elements of $\Phi(i, j')$ (cf. the Proposition) separate p and q . Therefore, p and q are at distance at least 2 in (the graph of) Σ , and if they are at distance 2, any point connected with both of them must belong to the kernel of every element of $\Phi(i, j')$, which implies $d_i = d_j (= d_{j'}) = 1$, by the Proposition. Conversely, if $d_i = d_j = 1$, the point $r \in \Sigma$ defined by

$$d(r) = a_k(r) = 0 \quad (k \neq i, j') \text{ and } a_i(r) > 0$$

is connected with both p and q because it is separated from them by no kernel of root. Indeed, if a root $\sum c_k a_k$ is strictly positive on r , one has $c_i > c_{j'}$ (because $a_i(r) + a_{j'}(r) = d(r) = 0$), hence $c_i = 1$, $c_{j'} = 0$ or $c_i = 0$, $c_{j'} = -1$, and the root $\sum c_k a_k$ takes positive values in both p and q , hence the corollary.

COROLLARY 2. *The distance between a vertex of Δ of type i and any opposite vertex is ≥ 3 ; it is equal to 3 if and only if there exists $j \in \{1, \dots, \hat{i}, \dots, l\}$ such that $d_i = d_j = 1$.*

This is an immediate consequence of the previous corollary.

From Corollary 2 and [LA], 1.3 (cf. condition (Q3)), we deduce:

COROLLARY 3. *Suppose that $d_i = 1$ for at most one value of i (i.e. that Σ is of type C_n, F_4, E_7 or E_8) and let Γ be an automorphism group of Δ whose orbits consist of pairwise opposite vertices. Then Δ/Γ is a geometry of type M (in the sense of [LA]), where M is the Coxeter matrix of Σ .*

EXAMPLE. Let K be a field, L a separable quadratic extension, Γ the Galois group, G a simple algebraic group, defined and anisotropic over K and whose relative Weyl group over L is of type $M = C_n, F_4, E_7$ or E_8 (e.g. $K = \mathbf{R}, L = \mathbf{C}$ and G is a compact group of type B_n, C_n, F_4, E_7 or E_8), and Δ the building of G over L . The proof of 4.7 in [GR] shows that if P, P' are two parabolic subgroups of G defined over L and permuted by the nontrivial element of Γ , the unipotent radical $R_u(P \cap P')$ is defined over K . Since G is K -anisotropic, it follows that $R_u(P \cap P')$ is trivial; in other words, $P \cap P'$ is reductive, which means that P and P' are opposite. Consequently, Δ/Γ is a geometry of type M , by Corollary 3.

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